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Change of basis, monomial relations, and P_t^s bases for the Steenrod algebra

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Abstract

The relationship between several common bases for the mod 2 Steenrod algebra is explored and a family of bases consisting of monomials in distinct P_t^s's is developed. A recursive change of basis formula is produced to convert between the Milnor basis and each of the bases for which the change of basis matrix in every grading is upper triangular. In particular, it is shown that the basis of admissible monomials, the P_t^s bases, and two bases due to Arnon, are all bases having this property, and the corresponding change of basis formula is produced for each of them. Some monomial relations for the mod 2 Steenrod algebra are then obtained by exploring the change of basis transformations. © 1998 Elsevier Science B.V.

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1. Introduction

There are many descriptions of bases for the mod 2 Steenrod algebra, A, in the literature. In addition to the classical basis of admissible monomials, there are the bases developed by Milnor [3] and Wall [4] as well as the more recent bases developed by Arnon [1] and Wood [5]. In this article we investigate the relationship between these bases, and add a family of bases consisting of monomials in distinct P_t^s's to the existing collection. These bases are all described in detail in Section 3.

Given so many different bases, a natural question to ask is: how can we convert from one basis to the other? Since almost all of the bases under consideration are described in terms of unevaluated products of A, such simple linear algebraic information actually can yield information about the product structure of A as well.

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All of the bases we consider can be described in terms of unevaluated monomials in Milnor basis elements. Thus it is a simple matter to convert from one of these bases, call it B, to the Milnor basis, B_{Mil}, by using the product formula (2) developed by Milnor [3]. A difficulty arises when trying to convert in the other direction: from the Milnor basis back to the basis B. Having such a formula for every basis would then allow us to convert between any two bases, indirectly, via the Milnor basis.

A brute force approach might be to compute the change of basis matrix, M, from B to B_{Mil} in a given grading using the Milnor product formula and compute M^{-1} to obtain the change of basis matrix in the opposite direction. But this approach is extremely inefficient and is unworkable in all but the lowest gradings where the vector space dimension is quite small.

Suppose, however, that we have the following situation.

Definition 1.1. Suppose there exists orderings, $\prec$ and $\dot{\prec}$, of bases B and B_{Mil} respectively, such that the change of basis matrix M (with respect to these orderings) is upper triangular in every grading. In this situation we say the basis B is *triangular* with respect to the Milnor basis.

This will be the situation if and only if there is an order preserving bijection γ: $B_{\text{Mil}} \to B$ such that $\gamma(\theta)$ is the $\prec$-largest summand of Milnor element θ when expressed in basis B.

Remark 1.1. If B is triangular with respect to the Milnor basis, then we have a well defined recursive formula to convert a Milnor basis element to the basis B. Namely, for any $\theta \in B_{\text{Mil}}$, we have

$$\theta_B = \gamma(\theta) + \sum_i (\theta_i)_B, \tag{1}$$

where x_B denotes the representation of x in basis B and $\gamma(\theta) = \theta + \sum_i \theta_i$ is the Milnor representation of $\gamma(\theta)_{\text{Mil}}$ obtained via the Milnor product formula.

This is a well-defined recursive formula because all of the Milnor basis elements θ_i must be strictly $\dot{\prec}$-less than θ and so the recursion must eventually end when we reach elements for which $\gamma(\theta) = \theta$ holds. Since A is finite dimensional in each grading, we must have $\gamma(\theta) = \theta$ for the θ which is the $\dot{\prec}$-smallest Milnor basis element in a given grading.

Thus in order to show that B is triangular with respect to the Milnor basis and determine the change of basis formula (1) for converting an element from the Milnor basis to basis B it suffices to:

1. Define a bijection $\gamma : B_{\text{Mil}} \to B$.
2. Define the ordering $\dot{\prec}$ on B_{Mil}. Then let $\prec$ be the unique ordering of B such that γ is order preserving.
3. Prove that $\gamma(\theta)$ is the $\prec$-largest summand of the representation of θ_B for any $\theta \in B_{\text{Mil}}$.

We will follow this procedure several times in what follows. Note that requirement 3 can also be satisfied by showing $\gamma^{-1}(\theta)$ is the $\dot{\prec}$-largest Milnor basis summand of the Milnor basis representation of θ for any $\theta \in B$, since γ is order preserving and the inverse of a triangular matrix is also triangular. Also in place of requirement 2 we can define the ordering $\prec$ on B and then let $\dot{\prec}$ be the unique ordering of B_{Mil} such that γ is order preserving.

In this article we will accomplish three things. First, we will construct a family of bases for the Steenrod algebra A consisting of monomials in distinct P^s_t's and add these bases to the list of bases being considered in this article. Second, we will determine which of the bases being considered are triangular with respect to the Milnor basis, and determine the change of basis formula of the form (1) for each basis that is. Finally, we will show how such information may lead to product information by determining an infinite family of elements which are both admissible monomial and Milnor basis elements.

2. Summary of main results

In this section we give a general overview of the main results which are contained in this paper. The details, notation, background and proofs will be presented later in the paper. Our first result is the construction of an infinite family of bases for A, using ideas similar to those discussed in [2, Ch. 15]. Let $\prec_R$ denote right lexicographic order (Definition 3.1).

Theorem 2.1. *The set, B_{PR}, of all monomials of the form $P^{s_0}_{t_0} P^{s_1}_{t_1} \cdots P^{s_p}_{t_p}$ such that $(s_0, t_0) \prec_R (s_1, t_1) \prec_R \cdots \prec_R (s_p, t_p)$ is a basis for A. In addition, any set B_P obtained by changing the order of the factors of any of the monomials in B_{PR} is also a basis for A.*

Adding these bases to the list of bases mentioned above (those of Wall, Arnon, and Wood and the basis of admissible monomials) we can completely determine which of these bases are triangular with respect to the Milnor basis and determine the change of basis formula of the form (1) by specifying the required γ.

Theorem 2.2. 1. *The following bases are triangular with respect to the Milnor basis and have change of basis formula* (1) *for the value of γ shown in Table* 1.

2. *Wall's basis, Wood's Y basis, and Wood's Z basis are not triangular with respect to the Milnor basis.*

It should be noted that in each case there is a simple heuristic for computing γ which makes these change of basis formulas quite easy to use in practice. We give both these heuristics and sample calculations along with the proofs in later sections of the article.

Table 1

Basis	γ required for formula (1)
Admissible monomials	Definition 4.1
Any P_t^s basis	Definition 5.1
Arnon C	Definition 6.1
Arnon A	Definition 7.1

Of some interest in its own right is the unusual ordering of the Milnor basis elements used in the proof of Theorem 2.2 for the Arnon A basis. This ordering is given in Definition 7.3.

One way to improve on the recursive change of basis formulas given in Theorem 2.2, would be to determine explicit non-recursive formulas. As a first step in this direction one might ask what elements two bases have in common. For example, it is well known that the $\mathrm{Sq}(n) = \mathrm{Sq}^n$ are common to both the Milnor and admissible monomial bases. Our final result determines an infinite family of elements which are common to these two bases. Let $\omega(n)$ be the smallest integer such that $2^{\omega(n)} > n$.

Theorem 2.3. *If* $r_i \equiv -1 \bmod 2^{\omega(r_{i+1})}$ *for all* $1 \le i < m$ *then* $\mathrm{Sq}(r_1, \ldots, r_m)$ *is an element of both the Milnor and admissible monomial bases. In this case* $\mathrm{Sq}(r_1, \ldots, r_m) = \mathrm{Sq}^{t_1}\mathrm{Sq}^{t_2} \cdots \mathrm{Sq}^{t_m}$ *where* $t_m = r_m$ *and* $t_i = r_i + 2t_{i+1}$ *for* $1 \le i < m$.

We point out that this linear algebra result is actually providing us with information about monomial products in A. Based on computer calculations we conjecture that these are the only elements common to the Milnor and admissible monomial bases. It is hoped that results of this sort would provide the first step in determining non-recursive change of basis formulas for these bases.

3. Bases for A: old and new

We begin by describing the bases to be discussed in this article. Algebraically the Steenrod algebra can be described as the quotient of the free associative graded algebra over the field with two elements, $\mathbb{F}_2$, on symbols Sq^n in grading n, by the ideal generated by the Adem relations:

$$\mathrm{Sq}^a\mathrm{Sq}^b = \sum_{n=0}^{\lfloor \frac{a}{2} \rfloor} \binom{b-n-1}{a-2n} \mathrm{Sq}^{a+b-n}\mathrm{Sq}^n \quad (\text{for } a < 2b),$$

where the binomial coefficients are taken mod 2 and $\mathrm{Sq}^0 = 1$, the multiplicative identity.

In order to describe the bases we wish to consider we first define the following.

Definition 3.1. Let $R = r_1, \ldots, r_m$ and $S = s_1, \ldots, s_n$ be finite sequences of non-negative integers. Define $r_k = 0$ for $k > m$ and $s_k = 0$ for $k > n$. Write $R \prec_R S$ if R is less than S

in lexicographic order from the right, i.e. if there exists i such that $r_i < s_i$ and $r_j = s_j$ for all $j > i$. If $R \prec_R S$ we will say R is rlex less than S. We make a similar definition for left lexicographic order, i.e. $R \prec_L S$ if there exists i such that $r_i < s_i$ and $r_j = s_j$ for all $j < i$ (where we take $r_k = 0$ for $k > m$ and $s_k = 0$ for $k > n$). If $R \prec_L S$ we will say R is llex less than S.

The bases we consider in this article are:

1. *Admissible monomials*: A monomial of the form $\mathrm{Sq}^{t_1}\mathrm{Sq}^{t_2}\cdots\mathrm{Sq}^{t_m}$ is said to be *admissible* if $t_i \geq 2t_{i+1}$ for $1 \leq i < m$. The set of all admissible monomials forms a basis for A which we will denote by B_{Adm}. Whether $\mathrm{Sq}^{t_1}\mathrm{Sq}^{t_2}\cdots\mathrm{Sq}^{t_m}$ is admissible or not, we will often abbreviate $\mathrm{Sq}^{t_1}\mathrm{Sq}^{t_2}\cdots\mathrm{Sq}^{t_m}$ by $\mathrm{Sq}^{t_1,\dots,t_m}$ and in addition if $T = t_1,\dots,t_m$ we will write $\mathrm{Sq}^{\langle T\rangle}$ for $\mathrm{Sq}^{t_1,\dots,t_m}$.

2. *Milnor* [3]: Milnor showed that A is also a Hopf algebra whose dual, A^*, is the polynomial algebra $\mathbb{F}_2\,[\xi_1,\xi_2,\dots]$ on generators ξ_n in grading 2^n-1. The basis of A which is dual to the basis of monomials in A^* is called the Milnor basis and will be denoted B_{Mil}. The element dual to $\xi_1^{r_1}\xi_2^{r_2}\cdots\xi_m^{r_m}$ in this basis is denoted $\mathrm{Sq}(r_1,\dots,r_m)$. Comparing with the notation given above we have $\mathrm{Sq}(n) = \mathrm{Sq}^n$. If $R = r_1,\dots,r_m$ is a finite sequence of non-negative integers, we will often use multi-index notation and write $\mathrm{Sq}\,\langle R\rangle$ for the Milnor basis element $\mathrm{Sq}(r_1,\dots,r_m)$.

The algebra structure on A in this basis can be described by the product formula given by Milnor. Namely,

$$\mathrm{Sq}(r_1,r_2,\dots)\mathrm{Sq}(s_1,s_2,\dots) = \sum_X \mathrm{Sq}(t_1,t_2,\dots), \tag{2}$$

where the sum is taken over all matrices $X = \langle x_{ij}\rangle$ satisfying:

$$\sum_i x_{ij} = s_j, \tag{3}$$

$$\sum_j 2^j x_{ij} = r_i, \tag{4}$$

$$\prod_h (x_{h0}, x_{h-1,1},\dots,x_{0h}) \equiv 1 \pmod 2. \tag{5}$$

where $(n_1,\dots,n_m)$ is the multinomial coefficient $(n_1+\cdots+n_m)!/(n_1!\cdots n_m!)$. (The value of x_{00} is never used and may be taken to be 0.) Each such allowable matrix produces a summand $\mathrm{Sq}(t_1,t_2,\dots)$ given by

$$t_h = \sum_{i+j=h} x_{ij}. \tag{6}$$

In such a situation we say that X *is a* $\mathrm{Sq}\,\langle R\rangle\,\mathrm{Sq}\,\langle S\rangle$*-allowable matrix which produces* $\mathrm{Sq}\,\langle T\rangle$. We will also find it convenient to say that X *produces the sequence* T if T satisfies (6) regardless of whether or not X is allowable.

3. *Arnon A* [1, Theorem 1A]: Define $X_k^n = \mathrm{Sq}^{2^n}\mathrm{Sq}^{2^{n-1}}\cdots\mathrm{Sq}^{2^k}$. Then the set of all monomials of the form $X_{k_0}^{n_0}X_{k_1}^{n_1}\cdots X_{k_p}^{n_p}$ such that $(n_0,k_0)\prec_L (n_1,k_1)\prec_L \cdots \prec_L (n_p,k_p)$ forms a basis for A which we will denoted by B_{ArA}.

4. *Wall* [4, p. 433]: Define $Q_k^n = \mathrm{Sq}^{2^k}\mathrm{Sq}^{2^{k+1}}\cdots\mathrm{Sq}^{2^n}$. Then the set of all monomials of the form $Q_{k_0}^{n_0}Q_{k_1}^{n_1}\cdots Q_{k_p}^{n_p}$ such that $(n_p,k_p)\prec_L (n_{p-1},k_{p-1})\prec_L \cdots \prec_L (n_0,k_0)$ forms a basis for A which we will denoted by B_{Wall}. This basis was also discussed in [1, Theorem 1B].

5. *Arnon C* [1, Theorem 1C]: A monomial of the form $\mathrm{Sq}^{t_m}\mathrm{Sq}^{t_{m-1}}\cdots\mathrm{Sq}^{t_1}$ is said to be *C-admissible* if $t_{i+1}\le 2t_i$ for $1\le i < m$ and t_i is divisible by 2^{i-1}. The set of all C-admissible monomials forms a basis for A which we will denote by B_{ArC}.

6. *Wood Y* [5, Theorem 1]: Define $Y_k^n = \mathrm{Sq}^{2^n(2^{k+1}-1)}$. Then the set of all monomials of the form $Y_{k_0}^{n_0}Y_{k_1}^{n_1}\cdots Y_{k_p}^{n_p}$ such that $(n_p,k_p)\prec_L (n_{p-1},k_{p-1})\prec_L \cdots \prec_L (n_0,k_0)$ forms a basis for A which we will denote by B_{WdY}. Wood shows that this basis has a nice property with respect to the Hopf subalgebras A_n of A generated by the Sq^{2^i} with $i\le n$. Namely if any factor of any summand of θ_{WdY} is not in A_n then θ itself is not in A_n.

7. *Wood Z* [5, Theorem 2]: Let $Y_k^n = \mathrm{Sq}^{2^n(2^{k+1}-1)}$ as above. Then the set of all monomials of the form $Y_{k_0}^{n_0}Y_{k_1}^{n_1}\cdots Y_{k_p}^{n_p}$ such that $(n_p+k_p,n_p)\prec_L (n_{p-1}+k_{p-1},n_{p-1})\prec_L \cdots \prec_L (n_0+k_0,n_0)$ forms a basis for A which we will denoted by B_{WdZ}. Wood shows that this basis also has the same nice property with respect to the Hopf subalgebras A_n that was mentioned above for the Y basis.

8. *P_t^s-bases*: In this article we will prove that the following is a basis for A. Let $P_t^s = \mathrm{Sq}(r_1,\ldots,r_t)$ where $r_t = 2^s$ and $r_i = 0$ for $i < t$. For each finite set, S, of P_t^s's choose an ordering of the elements of S, and let $M(S)$ be the monomial formed by taking the product of the elements of S in increasing order, i.e. if $S=\{P_{t_0}^{s_0},P_{t_1}^{s_1},\ldots,P_{t_p}^{s_p}\}$ and we order the elements of S in the order shown then $M(S)=P_{t_0}^{s_0}P_{t_1}^{s_1}\cdots P_{t_p}^{s_p}$. The monomials $M(S)$ form a basis for A. This gives us an infinite family of bases, one for each choice of ordering the sets S (not all of them are distinct, of course).

For example, the set of all monomials of the form $P_{t_0}^{s_0}P_{t_1}^{s_1}\cdots P_{t_p}^{s_p}$ such that $(s_0,t_0)\prec_R (s_1,t_1)\prec_R \cdots \prec_R (s_p,t_p)$ is one such basis which we will denote by B_{PR}.

Before leaving this section we give a few elementary definitions and notation that will be needed later on.

Definition 3.2. If B_{Name} is one of the bases of A described above and $\theta\in A$ then θ_{Name} will denote the representation of θ in that basis.

For example, $(\mathrm{Sq}^2\mathrm{Sq}^1)_{\mathrm{Mil}} = \mathrm{Sq}(3)+\mathrm{Sq}(0,1)$ while $\mathrm{Sq}(0,1)_{\mathrm{Adm}} = \mathrm{Sq}^2\mathrm{Sq}^1+\mathrm{Sq}^3$.

For any Milnor basis element, $\mathrm{Sq}(r_1,\ldots,r_m)$, it is clear from the definition that the grading or *degree* of $\mathrm{Sq}(r_1,\ldots,r_m)$ is $\sum_{i=1}^m(2^i-1)r_i$. For any of the other bases, the degree of a monomial is the sum of the degrees of its Milnor basis factors. The *excess* of $\mathrm{Sq}(r_1,\ldots,r_m)$ is $\sum_{i=1}^m r_i$ and its *length* is m. The excess of an admissible monomial $\mathrm{Sq}^{t_1,\ldots,t_m}$ is $t_m+\sum_{i=1}^{m-1}(t_i-2t_{i+1})$. We will denote the excess of $\theta\in B_{\mathrm{Mil}}$ by $\mathrm{ex}(\theta)$. Note that $\mathrm{Sq}(r_1,\ldots,r_m)$ is not uniquely determined by its degree, excess, and length as can be seen by the elements $\mathrm{Sq}(0,1,2,0,1)$ and $\mathrm{Sq}(2,0,0,1,1)$.

We can extend the definitions of left and right lexicographic order to both Milnor basis elements and monomials in Sq^n in the obvious manner, i.e. if $R \prec_R S$ then $\mathrm{Sq}\langle R\rangle \prec_R \mathrm{Sq}\langle S\rangle$ and $\mathrm{Sq}^R \prec_R \mathrm{Sq}^S$ and similarly if $R \prec_L S$ then $\mathrm{Sq}\langle R\rangle \prec_L \mathrm{Sq}\langle S\rangle$ and $\mathrm{Sq}^R \prec_L \mathrm{Sq}^S$.

For any positive integer n, let $\alpha_i(n)$ be the coefficient of 2^i in the binary expansion of n, i.e. $n=\sum_{i=0}^{\infty}\alpha_i(n)2^i$ and $\alpha_i(n)\in\{0,1\}$ for all i. We say that m and n are *disjoint* and write $m \asymp n$ if $\alpha_i(m)+\alpha_i(n)\leq 1$ for all i. It is well known that this is equivalent to the condition that the binomial coefficient $\binom{m+n}{m}$ is odd. Consequently, the multinomial coefficient $(n_1,\dots,n_m)$ is odd if and only if the integers $n_1,\dots,n_m$ are pairwise disjoint. This fact is used frequently throughout the article when evaluating condition (5).

We often write $2^i \in n$ for $\alpha_i(n)=1$ since the meaning is clear from the context. The following fact will be used implicitly several times and is an elementary exercise in binary arithmetic. Let $0\leq b < 2^t$. Then

$$2^l \in b \;\Leftrightarrow\; 2^l \in 2^t a + b \text{ and } l < t. \tag{7}$$

Finally, let $\nu(n)$ be the largest integer such that $n\equiv 0 \bmod 2^{\nu(n)}$ (and take $\nu(0)=\infty$). Let $\omega(n)$ be the smallest integer such that $2^{\omega(n)} > n$. Notice that for $n > 0$ we always have $2^{\nu(n)}\in n$ and also that $\nu(n) < \omega(n)$.

4. Milnor vs. admissible

We begin by focusing on the relationship between B_{Mil} and B_{Adm}. The elements $\mathrm{Sq}(n)$ $(=\mathrm{Sq}^n)$ are common to both the Milnor and admissible monomial bases. Therefore to express an admissible monomial in the Milnor basis, we only need use the product formula (2) for multiplying Milnor basis elements.

To convert a element from the Milnor basis to the basis of admissible monomials we now show that the basis of admissible monomials is triangular with respect to the Milnor basis and define the γ and ordering $\prec$ needed for the recursive formula (1). To satisfy requirement 1 following (1) we make the following definition.

Definition 4.1. Let $\mathrm{Sq}\langle R\rangle = \mathrm{Sq}(r_1,\dots,r_m)$ be a Milnor basis element. Define $\gamma(\mathrm{Sq}(r_1,\dots,r_m)) = \mathrm{Sq}^{t_1}\mathrm{Sq}^{t_2}\cdots\mathrm{Sq}^{t_m}$ where

$$t_i = \sum_{k=i}^{m} 2^{k-i} r_k. \tag{8}$$

(Abbreviation: we will sometimes write $\gamma\theta$ for $\gamma(\theta)$).

Note that the t_i can quickly be computed by starting with $t_m = r_m$ and then applying the simple recursion

$$t_i = r_i + 2t_{i+1}\,. \tag{9}$$

It follows immediately that $\gamma\,\mathrm{Sq}\,\langle R\rangle$ is an admissible monomial for any $\mathrm{Sq}\,\langle R\rangle \in B_{\mathrm{Mil}}$. The map γ is clearly a bijection on A in each degree and preserves both excess and rlex order. So we take both $\prec$ and $\dot{\prec}$ to be $\prec_R$ in this case to satisfy requirement 2 following (1). So in order to satisfy requirement 3 following (1) we show:

Theorem 4.1. *$\gamma\,\mathrm{Sq}\,\langle R\rangle$ is the rlex-largest summand of $\mathrm{Sq}\,\langle R\rangle_{\mathrm{Adm}}$.*

Hence B_{Adm} is triangular with respect to B_{Mil}. As a result we have a recursive formula of the form (1) for converting an element of A from the Milnor basis to the basis of admissible monomials.

Corollary 4.2. *Let $\mathrm{Sq}\,\langle R\rangle \in B_{\mathrm{Mil}}$ and suppose $\gamma(\mathrm{Sq}\,\langle R\rangle)_{\mathrm{Mil}} = \mathrm{Sq}\,\langle R\rangle + \sum_i \mathrm{Sq}\,\langle R_i\rangle$. Then*

$$\mathrm{Sq}\,\langle R\rangle_{\mathrm{Adm}} = \gamma\,\mathrm{Sq}\,\langle R\rangle + \sum_i \mathrm{Sq}\,\langle R_i\rangle_{\mathrm{Adm}}$$

is a well-defined recursive formula for computing $\mathrm{Sq}\,\langle R\rangle_{\mathrm{Adm}}$.

Note that $\gamma(\mathrm{Sq}\,\langle R\rangle)_{\mathrm{Mil}}$ can easily be obtained from the Milnor product formula (2). All of the elements $\mathrm{Sq}\,\langle R_i\rangle$ are strictly rlex-less than $\mathrm{Sq}\,\langle R\rangle$ which is why the recursive formula is well defined. This makes the formula quite easy to use in practice.

For example, to convert $\mathrm{Sq}(2,2)$ to the basis of admissible monomials using Corollary 4.2 we first compute $\gamma\,\mathrm{Sq}(2,2) = \mathrm{Sq}^6\mathrm{Sq}^2$. By the Milnor product formula, $\mathrm{Sq}(6)\,\mathrm{Sq}(2) = \mathrm{Sq}(2,2) + \mathrm{Sq}(5,1)$. The error term, $\mathrm{Sq}(5,1)$ is smaller than the original term $\mathrm{Sq}(2,2)$ in rlex order and we invoke Corollary 4.2 again. This time $\gamma\,\mathrm{Sq}(5,1) = \mathrm{Sq}^7\mathrm{Sq}^1$, but by the Milnor product formula we find that $\mathrm{Sq}(7)\,\mathrm{Sq}(1) = \mathrm{Sq}(5,1)$. Thus we have shown that

$$\mathrm{Sq}(2,2) = \mathrm{Sq}^6\mathrm{Sq}^2 + \mathrm{Sq}^7\mathrm{Sq}^1$$

which provides the conversion we desired.

In order to prove these results we begin by proving a useful lemma.

Let $\mathrm{Sq}(r_1,\ldots,r_m)$, $\mathrm{Sq}(s_1,\ldots,s_n) \in B_{\mathrm{Mil}}$. Let $X = \langle x_{ij}\rangle$ be the matrix

$$\begin{matrix} * & 0 & 0 & \cdots & 0 \\ r_1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{m-1} & 0 & 0 & \cdots & 0 \\ r_m - \sum 2^k s_k & s_1 & s_2 & \cdots & s_n \end{matrix}$$

We will call X the *rlex champion matrix for* $\mathrm{Sq}(r_1,\ldots,r_m)\,\mathrm{Sq}(s_1,\ldots,s_n)$.

Lemma 4.3. *If the rlex champion matrix for $\mathrm{Sq}\,\langle R\rangle\;\mathrm{Sq}\,\langle S\rangle$ produces T, then every other $\mathrm{Sq}\,\langle R\rangle\;\mathrm{Sq}\,\langle S\rangle$-allowable matrix produces a sequence which is rlex-less than T.*

Proof. By (3) and (4), any other such matrix must have $x_{ij} \neq 0$ for some $0 \leq i < m$ and $1 \leq j \leq n$. Let j be the largest such value. Then by (3) we have $x_{mj} < s_j$. Therefore the element $\mathrm{Sq}(u_1, \ldots, u_{m+n})$ produced by the new matrix must have $u_k = s_k = t_k$ for $m + j < k \leq m + n$ and $u_{m+j} = x_{mj} < s_j = t_j$. Thus $\mathrm{Sq}\langle U \rangle \prec_R \mathrm{Sq}\langle T \rangle$. □

As an immediate consequence we have:

Corollary 4.4. *Let T be the sequence produced by the rlex champion matrix for $\mathrm{Sq}\langle R \rangle \mathrm{Sq}\langle S \rangle$. If $U \prec_R S$ then for every Milnor summand $\mathrm{Sq}\langle V \rangle$ of the product $\mathrm{Sq}\langle R \rangle \mathrm{Sq}\langle U \rangle$ we have $V \prec_R T$.*

This follows from Lemma 4.3 and the fact that the sequence T' produced by the rlex champion matrix for $\mathrm{Sq}\langle R \rangle \mathrm{Sq}\langle U \rangle$ is easily seen to be rlex-less than T.

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. We wish to show that $\gamma \mathrm{Sq}\langle R \rangle$ is the rlex-largest summand of $\mathrm{Sq}\langle R \rangle_{\mathrm{Adm}}$. Since γ is bijective and preserves rlex order, it suffices to show that for any admissible monomial $\mathrm{Sq}^{\langle T \rangle}$ the Milnor basis element $\gamma^{-1}(\mathrm{Sq}^{\langle T \rangle})$ is the rlex-largest summand of $(\mathrm{Sq}^{\langle T \rangle})_{\mathrm{Mil}}$.

Let $T = t_1, \ldots, t_m$ be an admissible sequence. Recall that by (9) $\gamma^{-1}(\mathrm{Sq}^{\langle T \rangle}) = \mathrm{Sq}(r_1, \ldots, r_m)$ where $r_m = t_m$ and $r_i = t_i - 2t_{i+1}$ for $i < m$. We proceed by induction on m.

If $m = 1$ then $(\mathrm{Sq}^{\langle T \rangle})_{\mathrm{Mil}} = \mathrm{Sq}(t_1)$ and $\gamma^{-1}(\mathrm{Sq}^{\langle T \rangle}) = \mathrm{Sq}(t_1)$, so the base case holds.

Now for the inductive hypothesis assume that for any admissible monomial $\mathrm{Sq}^{\langle S \rangle}$ of length less than m, $\gamma^{-1}(\mathrm{Sq}^{\langle S \rangle})$ is the rlex-largest summand of $(\mathrm{Sq}^{\langle S \rangle})_{\mathrm{Mil}}$. Then in particular, $\gamma^{-1}(\mathrm{Sq}^{t_2,\ldots,t_m}) = \mathrm{Sq}(r_2, \ldots, r_m)$ is the rlex-largest summand of $(\mathrm{Sq}^{t_2,\ldots,t_m})_{\mathrm{Mil}}$. So we can write

$$(\mathrm{Sq}^{t_2,\ldots,t_m})_{\mathrm{Mil}} = \mathrm{Sq}(r_2, \ldots, r_m) + \sum \mathrm{Sq}\langle R_i \rangle,$$

where $\mathrm{Sq}\langle R_i \rangle \prec_R \mathrm{Sq}(r_2, \ldots, r_m)$ for all i.

Thus we have

$$\begin{aligned}(\mathrm{Sq}^{t_1,\ldots,t_m})_{\mathrm{Mil}} &= \mathrm{Sq}(t_1)\,(\mathrm{Sq}^{t_2,\ldots,t_m})_{\mathrm{Mil}} \\ &= \mathrm{Sq}(t_1)\left(\mathrm{Sq}(r_2, \ldots, r_m) + \sum \mathrm{Sq}\langle R_i \rangle\right) \\ &= \mathrm{Sq}(t_1)\,\mathrm{Sq}(r_2, \ldots, r_m) + \sum \mathrm{Sq}(t_1)\,\mathrm{Sq}\langle R_i \rangle.\end{aligned}$$

The rlex champion matrix X for $\mathrm{Sq}(t_1)\,\mathrm{Sq}(r_2, \ldots, r_m)$ is

$$\begin{matrix} * & 0 & 0 & \cdots & 0 \\ t_1 - \sum_{k=2}^{m} 2^{k-1} r_k & r_2 & r_3 & \cdots & r_m. \end{matrix}$$

Clearly X is admissible. To see that it produces $\mathrm{Sq}(r_1,\ldots,r_m)$ we need only verify that

$$t_1 - \sum_{k=2}^{m} 2^{k-1} r_k = t_1 - 2\sum_{k=2}^{m} 2^{k-2} r_k$$
$$= t_1 - 2t_2$$
$$= r_1.$$

Therefore by Lemma 4.3 every other $\mathrm{Sq}(t_1)\,\mathrm{Sq}(r_2,\ldots,r_m)$-allowable matrix produces Milnor elements which are rlex-less than $\mathrm{Sq}(r_1,\ldots,r_m)$. So $\mathrm{Sq}(r_1,\ldots,r_m)$ is the rlex-largest summand of $\mathrm{Sq}(t_1)\,\mathrm{Sq}(r_2,\ldots,r_m)$. In addition, every summand of $\mathrm{Sq}(t_1)\,\mathrm{Sq}\langle R_i\rangle$ is rlex less than $\mathrm{Sq}(r_1,\ldots,r_m)$ by Corollary 4.4. □

5. Milnor vs. P_t^s

We now turn our attention to the relationship between B_{Mil} and B_{PR}. We use methods and ideas similar to those discussed in [2, Chapter 15].

All of the results and arguments in this section carry over to any P_t^s basis, but we illustrate them for this particular ordering of the monomial factors. The elements P_t^s are common to both the Milnor and P_t^s bases. Therefore to express an element of B_{PR} in the Milnor basis, we only need use the product formula (2) for multiplying Milnor basis elements.

Notice that we have not yet shown that B_{PR} is a basis for A although we have defined it as a set. To see that B_{PR} is in fact a triangular basis with respect to the Milnor basis we begin by defining a grading preserving bijection $\gamma : B_{\mathrm{Mil}} \to B_{PR}$.

Definition 5.1. Let $\mathrm{Sq}(r_1,\ldots,r_m) \in B_{\mathrm{Mil}}$. Define

$$\gamma(\mathrm{Sq}(r_1,\ldots,r_m)) = P_{t_0}^{s_0} P_{t_1}^{s_1} \cdots P_{t_p}^{s_p}$$

where the right hand side is the unique monomial in B_{PR} satisfying

$$P_j^i \text{ is a factor of } P_{t_0}^{s_0} P_{t_1}^{s_1} \cdots P_{t_p}^{s_p} \iff \alpha_i(r_j) = 1$$

for all i and j.

The map γ is clearly a bijection on A in each grading.

There is a useful heuristic device for computing the $\gamma\,\mathrm{Sq}(r_1,\ldots,r_m)$. We define the *binary chart of* $\mathrm{Sq}(r_1,\ldots,r_m)$ to be the array:

$$
\begin{array}{cc|cccc}
 & \vdots & \vdots & \vdots & \vdots & \\
 & 2 & \alpha_2(r_1) & \alpha_2(r_2) & \alpha_2(r_3) & \cdots \\
s & 1 & \alpha_1(r_1) & \alpha_1(r_2) & \alpha_1(r_3) & \cdots \\
 & 0 & \alpha_0(r_1) & \alpha_0(r_2) & \alpha_0(r_3) & \cdots \\
\hline
 & & 1 & 2 & 3 & \cdots \\
 & & & t & &
\end{array}
$$

In other words simply write the binary expansions of the numbers $r_1,\ldots,r_m$ vertically next to each other. Then P_t^s is a factor of $\gamma\,\mathrm{Sq}(r_1,\ldots,r_m)$ if and only if there is a 1 in location (s,t) in the binary chart. The factors are then multiplied in the correct order for whichever P_t^s basis we are considering.

For example, to compute $\gamma\,\mathrm{Sq}(2,5,1)$ we make the chart:

$$
\begin{array}{ccc}
 & 1 & \\
1 & 0 & \\
0 & 1 & 1
\end{array}
$$

and read off the factors P_1^1, P_2^0, P_2^2, and P_3^0. Multiplying them in the correct order for B_{PR} we get $\gamma\mathrm{Sq}(2,5,1) = P_1^1P_2^0P_2^2P_3^0$.

Now define an ordering $\prec_E$ on B_{Mil} as follows.

Definition 5.2. For any $\mathrm{Sq}\langle R\rangle, \mathrm{Sq}\langle S\rangle \in B_{\mathrm{Mil}}$, we say $\mathrm{Sq}\langle R\rangle \prec_E \mathrm{Sq}\langle S\rangle$ if

$$\mathrm{ex}(\mathrm{Sq}\langle R\rangle) < \mathrm{ex}(\mathrm{Sq}\langle S\rangle)$$

or else

$$\mathrm{ex}(\mathrm{Sq}\langle R\rangle) = \mathrm{ex}(\mathrm{Sq}\langle S\rangle) \text{ and } \mathrm{Sq}\langle R\rangle \prec_R \mathrm{Sq}\langle S\rangle.$$

The second condition is simply used to make a total ordering out of the partial ordering induced by excess and is never used.

Finally let $\prec$ be the ordering induced on B_{PR} induced by the bijection γ and $\prec_E$. Then we have:

Theorem 5.1. *$\gamma\,\mathrm{Sq}\langle R\rangle$ is the $\prec$-largest summand of* $\mathrm{Sq}\langle R\rangle_{PR}$.

It follows immediately that the elements of B_{PR} are linearly independent in each grading and since γ is a bijection, B_{PR} is, indeed, a basis as claimed. Further, with this definition of γ and $\prec_E$ we have satisfied requirements 1–3 in Section 1 and so B_{PR} is triangular with respect to B_{Mil}. As a result we have a recursive formula of the form (1) for converting an element of A from the Milnor basis to the basis of admissible monomials.

Corollary 5.2. *Let* $\mathrm{Sq}\langle R\rangle \in B_{\mathrm{Mil}}$ *and suppose* $\gamma(\mathrm{Sq}\langle R\rangle)_{\mathrm{Mil}} = \mathrm{Sq}\langle R\rangle + \sum_i \mathrm{Sq}\langle R_i\rangle$. *Then*

$$\mathrm{Sq}\langle R\rangle_{PR} = \gamma\,\mathrm{Sq}\langle R\rangle + \sum_i \mathrm{Sq}\langle R_i\rangle_{PR}$$

is a well-defined recursive formula for computing $\mathrm{Sq}\langle R\rangle_{PR}$.

Note that $\gamma(\mathrm{Sq}\langle R\rangle)_{\mathrm{Mil}}$ can easily be obtained from the Milnor product formula (2). All of the elements $\mathrm{Sq}\langle R_i\rangle$ are strictly $\prec_E$-less than $\mathrm{Sq}\langle R\rangle$ which is why the recursive formula is well defined.

For example, to convert $\mathrm{Sq}(4,2)$ to the basis B_{PR} using Corollary 5.2 we first compute $\gamma\,\mathrm{Sq}(4,2) = P_1^2 P_2^1$. By the Milnor product formula, $P_1^2 P_2^1 = \mathrm{Sq}(4)\,\mathrm{Sq}(0,2) = \mathrm{Sq}(4,2) + \mathrm{Sq}(0,1,1)$. The error term, $\mathrm{Sq}(0,1,1)$ is smaller than the original term $\mathrm{Sq}(4,2)$ in $\prec_E$ order and so we invoke Corollary 5.2 again. This time $\gamma\,\mathrm{Sq}(0,1,1) = P_2^0 P_3^0$, but by the Milnor product formula we find that $P_2^0 P_3^0 = \mathrm{Sq}(0,1)\,\mathrm{Sq}(0,0,1) = \mathrm{Sq}(0,1,1)$. Thus we have shown that

$$\mathrm{Sq}(4,2) = P_1^2 P_2^1 + P_2^0 P_3^0$$

which provides the conversion we desired.

In order to prove these results we begin by proving a few useful lemmas.

Let $\mathrm{Sq}(r_1,\ldots,r_m)$, $\mathrm{Sq}(s_1,\ldots,s_n) \in B_{\mathrm{Mil}}$. Let $X = \langle x_{ij}\rangle$ be the matrix

$$\begin{array}{ccccc} * & s_1 & s_2 & \cdots & s_n \\ r_1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_m & 0 & 0 & \cdots & 0 \end{array}$$

We will call X the *excess champion matrix for* $\mathrm{Sq}(r_1,\ldots,r_m)\ \mathrm{Sq}(s_1,\ldots,s_n)$.

Let $R = r_1, r_2, \ldots$ and $S = s_1, s_2, \ldots$. Then we define the obvious sum

$$R + S = r_1 + s_1, r_2 + s_2, \ldots, r_i + s_i, \ldots.$$

In this notation we see that the excess champion matrix for $\mathrm{Sq}\langle R\rangle\mathrm{Sq}\langle S\rangle$ produces $\mathrm{Sq}\langle R+S\rangle$. Notice that $\mathrm{ex}(\mathrm{Sq}\langle R+T\rangle) = \mathrm{ex}(\mathrm{Sq}\langle R\rangle) + \mathrm{ex}(\mathrm{Sq}\langle T\rangle)$.

Lemma 5.3. *If* X *is an allowable* $\mathrm{Sq}\langle R\rangle\ \mathrm{Sq}\langle S\rangle$ *matrix which produces* $\mathrm{Sq}\langle T\rangle$ *then* $\mathrm{ex}(\mathrm{Sq}\langle T\rangle) < \mathrm{ex}(\mathrm{Sq}\langle R+S\rangle)$.

Proof. Since the excess of $\mathrm{Sq}\langle T\rangle = \mathrm{Sq}(t_1,t_2,\ldots)$ is $\sum t_i$ and by (6) each t_i is the sum of the ith diagonal of $X = \langle x_{ij}\rangle$, it follows that $\mathrm{ex}(\mathrm{Sq}\langle T\rangle) = \sum_{i,j} x_{ij}$, i.e. it is the sum of all of the entries of the matrix. By (3) the sum of the entries in columns to the right of column 0, $\sum_{j>0} x_{ij}$ must equal $\mathrm{ex}(\mathrm{Sq}\langle S\rangle)$. By (4) $x_{i0} \le r_i$ for each i so that the entries in column 0 must have a sum less than or equal to the excess of $\mathrm{Sq}\langle R\rangle$, i.e. $\sum_{j=0} x_{ij} \le \mathrm{ex}(\mathrm{Sq}\langle R\rangle)$. But since X is not the excess champion matrix, we must have $x_{uv} \ne 0$ for some $u > 0$ and $v > 0$. But by (4) it follows that $x_{u0} < r_u$ and so the sum of column 0 is strictly less than $\mathrm{ex}(\mathrm{Sq}\langle R\rangle)$. Hence,

$$\begin{aligned}\mathrm{ex}(\mathrm{Sq}\langle T\rangle) &= \sum_{i,j} x_{ij}\\ &= \sum_{j=0} x_{i,j} + \sum_{j>0} x_{i,j}\\ &< \mathrm{ex}(\mathrm{Sq}\langle R\rangle) + \mathrm{ex}(\mathrm{Sq}\langle S\rangle)\\ &= \mathrm{ex}(\mathrm{Sq}\langle R+S\rangle)\end{aligned}$$

as claimed. □

As an immediate consequence we have

Corollary 5.4. *If* $\mathrm{ex}(\mathrm{Sq}\langle U\rangle) < \mathrm{ex}(\mathrm{Sq}\langle R\rangle)$ *then every Milnor summand* $\mathrm{Sq}\langle T\rangle$ *of the product* $\mathrm{Sq}\langle U\rangle\mathrm{Sq}\langle S\rangle$ *(or* $\mathrm{Sq}\langle S\rangle\mathrm{Sq}\langle U\rangle$*) has excess less than* $\mathrm{ex}(\mathrm{Sq}\langle R+S\rangle)$.

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. We wish to show that $\gamma\,\mathrm{Sq}\langle R\rangle$ is the $\prec$-largest summand of $\mathrm{Sq}\langle R\rangle_{PR}$. It suffices to show that for any element $P_{t_0}^{s_0}P_{t_1}^{s_1}\cdots P_{t_p}^{s_p}$ in B_{PR}, $\gamma^{-1}(P_{t_0}^{s_0}P_{t_1}^{s_1}\cdots P_{t_p}^{s_p})$ is the $\prec_E$-largest summand of $\left(P_{t_0}^{s_0}P_{t_1}^{s_1}\cdots P_{t_p}^{s_p}\right)_{\mathrm{Mil}}$. We will show something slightly stronger, namely that $\gamma^{-1}(P_{t_0}^{s_0}P_{t_1}^{s_1}\cdots P_{t_p}^{s_p})$ is a summand of $\left(P_{t_0}^{s_0}P_{t_1}^{s_1}\cdots P_{t_p}^{s_p}\right)_{\mathrm{Mil}}$ and every other Milnor summand of $\left(P_{t_0}^{s_0}P_{t_1}^{s_1}\cdots P_{t_p}^{s_p}\right)_{\mathrm{Mil}}$ will have excess strictly less than $\mathrm{ex}\left(\gamma^{-1}(P_{t_0}^{s_0}P_{t_1}^{s_1}\cdots P_{t_p}^{s_p})\right)$.

Let $\theta = P_{t_0}^{s_0}P_{t_1}^{s_1}\cdots P_{t_p}^{s_p} \in B_{PR}$ and let $\mathrm{Sq}(r_1,\ldots,r_m) = \gamma^{-1}(P_{t_0}^{s_0}P_{t_1}^{s_1}\cdots P_{t_p}^{s_p})$ where $r_i = \sum_{t_j=i} 2^{s_j}$. We proceed by induction on p .

If $p=0$ then $(P_{t_0}^{s_0})_{\mathrm{Mil}} = P_{t_0}^{s_0}$ and $\gamma^{-1}(P_{t_0}^{s_0}) = P_{t_0}^{s_0}$, so the base case holds.

Now for the inductive hypothesis assume that for any element $\theta \in B_{PR}$ having fewer than $p+1$ factors, $\gamma^{-1}(\theta)$ is the rlex-largest summand of θ_{Mil}. Then in particular,

$$\gamma^{-1}\left(P_{t_0}^{s_0}P_{t_1}^{s_1}\cdots P_{t_{p-1}}^{s_{p-1}}\right) = \mathrm{Sq}(r_1,\ldots,r_{m-1},r_m-2^{s_p})$$

is a summand of $\left(P_{t_0}^{s_0}P_{t_1}^{s_1}\cdots P_{t_p}^{s_p}\right)_{\mathrm{Mil}}$ and every other summand has excess less than $\mathrm{ex}(\mathrm{Sq}(r_1,\ldots,r_{m-1},r_m-2^{s_p}))$. So we can write

$$\left(P_{t_0}^{s_0}P_{t_1}^{s_1}\cdots P_{t_p}^{s_p}\right)_{\mathrm{Mil}} = \mathrm{Sq}(r_1,\ldots,r_{m-1},r_m-2^{s_p}) + \sum \mathrm{Sq}\langle R_i\rangle,$$

where $\mathrm{ex}(\mathrm{Sq}\langle R_i\rangle) < \mathrm{ex}(\mathrm{Sq}(r_1,\ldots,r_{m-1},r_m-2^{s_p}))$ for all i.

Thus we have

$$\begin{aligned}\left(P_{t_0}^{s_0}P_{t_1}^{s_1}\cdots P_{t_p}^{s_p}\right)_{\mathrm{Mil}} &= \left(P_{t_0}^{s_0}P_{t_1}^{s_1}\cdots P_{t_{p-1}}^{s_{p-1}}\right)_{\mathrm{Mil}} P_{t_p}^{s_p}\\ &= \left(\mathrm{Sq}(r_1,\ldots,r_{m-1},r_m-2^{s_p}) + \sum \mathrm{Sq}\langle R_i\rangle\right) P_{t_p}^{s_p}\\ &= \mathrm{Sq}(r_1,\ldots,r_{m-1},r_m-2^{s_p})P_{t_p}^{s_p} + \sum \mathrm{Sq}\langle R_i\rangle P_{t_p}^{s_p}.\end{aligned}$$

Now each of the Milnor summands of $\sum \mathrm{Sq}\langle R_i\rangle P_{t_p}^{s_p}$ must have excess strictly less than $\mathrm{ex}(\mathrm{Sq}(r_1,\ldots,r_m))$ by Corollary 5.4. Every summand of $\mathrm{Sq}(r_1,\ldots,r_{m-1},r_m-2^{s_p})P_{t_p}^{s_p}$ other than $\mathrm{Sq}(r_1,\ldots,r_m)$ must have excess strictly less than $\mathrm{ex}(\mathrm{Sq}(r_1,\ldots,r_m))$ by Lemma 5.3. Finally, it is easy to see that the excess champion matrix associated with $\mathrm{Sq}(r_1,\ldots,r_{m-1},r_m-2^{s_p})P_{t_p}^{s_p}$ is allowable and thus $\mathrm{Sq}(r_1,\ldots,r_m)$ is a summand of $(P_{t_0}^{s_0}P_{t_1}^{s_1}\cdots P_{t_p}^{s_p})_{\mathrm{Mil}}$. □

6. Milnor vs. Arnon C

We now turn to the relationship between B_{Mil} and B_{ArC}. In many ways this relationship is similar to the situation we find for B_{Adm}. The elements $\mathrm{Sq}(n)$ are again common to both bases, so to express $\theta \in B_{\mathrm{ArC}}$ in the Milnor basis, we only need use the product formula (2).

To convert a element from the Milnor basis to the Arnon C basis we follow the now familiar path of showing that the basis of C-admissible monomials is triangular with respect to the Milnor basis by defining the appropriate γ and ordering $\prec$ needed for the recursive formula of the form (1).

Definition 6.1. Let $\mathrm{Sq}\langle R\rangle = \mathrm{Sq}(r_1,\ldots,r_m)$ be a Milnor basis element. Define $\gamma(\mathrm{Sq}(r_1,\ldots,r_m)) = \mathrm{Sq}^{t_m}\mathrm{Sq}^{t_{m-1}}\cdots\mathrm{Sq}^{t_1}$, where

$$t_i = 2^{i-1}\sum_{k=i}^{m} r_k. \tag{10}$$

Note that $\gamma(\mathrm{Sq}(r_1,\ldots,r_m))$ can easily be computed by the following heuristic. First, write the sequence $r_1,\ldots,r_m$ in a vertical column with r_1 on top. Then working to the left, construct the following triangular shaped diagram in which each column contains entries which are twice the entry to its right:

$$\begin{array}{ccccc}
 & & & & r_1 \\
 & & & 2r_2 & r_2 \\
 & & & \vdots & \vdots \\
 & 2^{m-2}r_{m-1} & \cdots & 2r_{m-1} & r_{m-1} \\
2^{m-1}r_m & 2^{m-2}r_m & \cdots & 2r_m & r_m \\
\hline
t_m & t_{m-1} & \cdots & t_2 & t_1
\end{array}$$

the value of t_i is then simply the sum of the ith column from the right as indicated.

It is clear from the definition that t_i is divisible by $2^{i-1}$and also that

$$t_{i+1} = 2t_i - 2^i r_i \tag{11}$$

holds for $1 \le i < m$. Hence $t_{i+1} \le 2t_i$ so that $\gamma\,\mathrm{Sq}\langle R\rangle$ is indeed in B_{ArC}.

The map γ is a bijection on A in each grading and by (11)

$$\gamma^{-1}\left(\mathrm{Sq}^{t_m,\dots,t_1}\right) = \mathrm{Sq}(r_1,\dots,r_m)$$

where $r_i = (2t_i - t_{i+1})/2^i$ for $1 \le i < m$ and $r_m = (t_m/2^{m-1})$.

For this basis we choose $\prec_R$ for the ordering of B_{Mil} and let $\prec$ be the ordering induced by γ on B_{ArC}. Then we have

Theorem 6.1. *$\gamma\mathrm{Sq}\langle R\rangle$ is the $\prec$-largest summand of $\mathrm{Sq}\langle R\rangle_{\mathrm{ArC}}$.*

Hence B_{ArC} is triangular with respect to B_{Mil} and we have a recursive formula of the form (1) for converting an element of A from the Milnor basis to the basis of C-admissible monomials.

Corollary 6.2. *Let $\mathrm{Sq}\langle R\rangle \in B_{\mathrm{Mil}}$ and suppose $\gamma(\mathrm{Sq}\langle R\rangle)_{\mathrm{Mil}} = \mathrm{Sq}\langle R\rangle + \sum_i \mathrm{Sq}\langle R_i\rangle$. Then*

$$\mathrm{Sq}\langle R\rangle_{\mathrm{ArC}} = \gamma\,\mathrm{Sq}\langle R\rangle + \sum_i \mathrm{Sq}\langle R_i\rangle_{\mathrm{ArC}}$$

is a well-defined recursive formula for computing $\mathrm{Sq}\langle R\rangle_{\mathrm{ArC}}$.

Note that once again $\gamma(\mathrm{Sq}\langle R\rangle)_{\mathrm{Mil}}$ can easily be obtained from the Milnor product formula (2) and all of the elements $\mathrm{Sq}\langle R_i\rangle$ are strictly rlex-less than $\mathrm{Sq}\langle R\rangle$ which is why the recursive formula is well defined.

For example, to convert $\mathrm{Sq}(3,2)$ to the basis of C-admissible monomials using Corollary 6.2 we first compute $\gamma\mathrm{Sq}(3,2) = \mathrm{Sq}^4\mathrm{Sq}^5$. By the Milnor product formula, $\mathrm{Sq}(4)\mathrm{Sq}(5) = \mathrm{Sq}(3,2) + \mathrm{Sq}(6,1)$. The error term, $\mathrm{Sq}(6,1)$ is smaller than the original term $\mathrm{Sq}(3,2)$ in rlex order and so we invoke Corollary 6.2 again. This time $\gamma\mathrm{Sq}(6,1) = \mathrm{Sq}^2\mathrm{Sq}^7$, but by the Milnor product formula we find that $\mathrm{Sq}(2)\mathrm{Sq}(7) = \mathrm{Sq}(6,1)$. Thus we have shown that

$$\mathrm{Sq}(3,2) = \mathrm{Sq}^4\mathrm{Sq}^5 + \mathrm{Sq}^2\mathrm{Sq}^7$$

which provides the conversion we desired.

We are now ready to prove Theorem 6.1.

Proof of Theorem 6.1. Once again it suffices to show that for any C-admissible monomial $\mathrm{Sq}^{\langle T\rangle}$ the Milnor basis element $\gamma^{-1}\left(\mathrm{Sq}^{\langle T\rangle}\right)$ is the rlex-largest summand of $\left(\mathrm{Sq}^{\langle T\rangle}\right)_{\mathrm{Mil}}$.

Let $\mathrm{Sq}^{\langle T\rangle} = \mathrm{Sq}^{t_m,\dots,t_1}$ be a C-admissible monomial. Then

$$\gamma^{-1}\left(\mathrm{Sq}^{t_m,\dots,t_1}\right) = \mathrm{Sq}(r_1,\dots,r_m)$$

where $r_i = (2t_i - t_{i+1})/2^i$ for $1 \le i < m$ and $r_m = (t_m/2^{m-1})$. We proceed by induction on m.

If $m = 1$ then $\left(\mathrm{Sq}^{\langle T\rangle}\right)_{\mathrm{Mil}} = \mathrm{Sq}(t_1)$ and $\gamma^{-1}\left(\mathrm{Sq}^{\langle T\rangle}\right) = \mathrm{Sq}(t_1)$, so the base case holds.

Now for the inductive hypothesis assume that for any admissible monomial $\mathrm{Sq}^{\langle S\rangle}$ of length less than m, $\gamma^{-1}(\mathrm{Sq}^{\langle S\rangle})$ is the rlex-largest summand of $(\mathrm{Sq}^{\langle S\rangle})_{\mathrm{Mil}}$. Then in particular, we can compute

$$\gamma^{-1}\left(\mathrm{Sq}^{t_{m-1},\dots,t_1}\right) = \mathrm{Sq}\,(r_1,\dots,r_{m-2},r_{m-1}+r_m)$$

which must be the rlex-largest summand of $\left(\mathrm{Sq}^{t_{m-1},\dots,t_1}\right)_{\mathrm{Mil}}$. So we can write

$$\left(\mathrm{Sq}^{t_m,\dots,t_1}\right)_{\mathrm{Mil}} = \mathrm{Sq}\,(r_1,\dots,r_{m-2},r_{m-1}+r_m) + \sum \mathrm{Sq}\langle R_i\rangle$$

where $\mathrm{Sq}\langle R_i\rangle \prec_R \mathrm{Sq}\,(r_1,\dots,r_{m-2},r_{m-1}+r_m)$ for all i.

Thus we have

$$\begin{aligned}\left(\mathrm{Sq}^{t_m,\dots,t_1}\right)_{\mathrm{Mil}} &= \mathrm{Sq}\,(t_m)\left(\mathrm{Sq}^{t_{m-1},\dots,t_1}\right)_{\mathrm{Mil}}\\ &= \mathrm{Sq}\,(t_m)(\mathrm{Sq}\,(r_1,\dots,r_{m-2},r_{m-1}+r_m)+\sum \mathrm{Sq}\langle R_i\rangle)\\ &= \mathrm{Sq}\,(t_m)\,\mathrm{Sq}\,(r_1,\dots,r_{m-2},r_{m-1}+r_m)+\sum \mathrm{Sq}\,(t_m)\,\mathrm{Sq}\langle R_i\rangle.\end{aligned}$$

The rlex champion matrix for $\mathrm{Sq}\,(t_m)\,\mathrm{Sq}\,(r_1,\dots,r_{m-2},r_{m-1}+r_m)$ is not allowable in this case so instead we let X be the matrix

$$\begin{matrix} * & r_1 & \cdots & r_{m-2} & r_{m-1}\\ 0 & 0 & \cdots & 0 & r_m. \end{matrix}$$

Clearly X is allowable and produces $\mathrm{Sq}\,(r_1,\dots,r_m)$. To see that X is indeed a $\mathrm{Sq}\,(t_m)$ $\mathrm{Sq}\,(r_1,\dots,r_{m-2},r_{m-1}+r_m)$ matrix we need only note that $r_m = (t_m/2^{m-1})$.

Now every other $\mathrm{Sq}\,(t_m)\,\mathrm{Sq}\,(r_1,\dots,r_{m-2},r_{m-1}+r_m)$-allowable matrix produces Milnor elements which are rlex-less than $\mathrm{Sq}\,(r_1,\dots,r_m)$ since by (4) any other such matrix must produce $\mathrm{Sq}\,(t_1,\dots,t_m)$ with $t_m < r_m$.

So it remains to show that every summand of $\sum \mathrm{Sq}\,(t_m)\,\mathrm{Sq}\langle R_i\rangle$ is rlex-less than $\mathrm{Sq}\,(r_1,\dots,r_m)$. Let $\mathrm{Sq}\langle R_i\rangle = \mathrm{Sq}\,(u_1,\dots,u_n)$ be one of the summands. We know $\mathrm{Sq}\langle R_i\rangle$ is rlex-less than $\mathrm{Sq}\,(r_1,\dots,r_{m-2},r_{m-1}+r_m)$. If $n < m-1$ then every summand of $\mathrm{Sq}\,(t_m)\,\mathrm{Sq}\langle R_i\rangle$ has length less than m and is therefore rlex-less than $\mathrm{Sq}\,(r_1,\dots,r_m)$. On the other hand, if $n = m-1$ then there exists j such that $u_j < r_j$ and

$$\mathrm{Sq}\,(u_1,\dots,u_{m-1}) = \mathrm{Sq}\,\left(u_1,\dots,u_j,r_{j+1},\dots,r_{m-2},r_{m-1}+r_m\right).$$

In this case, the matrix

$$\begin{matrix} * & u_1 & \cdots & u_j & r_{j+1} & \cdots & r_{m-2} & r_{m-1}\\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & r_m \end{matrix}$$

produces the sequence $\left(u_1,\dots,u_j,r_{j+1},\dots,r_{m-1},r_m\right)$ (whether or not it is allowable) which is clearly rlex-less than $(r_1,\dots,r_m)$. By the same argument as above, any other $\mathrm{Sq}\,(t_m)\,\mathrm{Sq}\langle R_i\rangle$-allowable matrix must produce $\mathrm{Sq}\langle V\rangle$ for which $V \prec_R U$ which is in turn rlex-less than $(r_1,\dots,r_m)$. □

7. Milnor vs. Arnon A

The strangest of the bases discussed here which are triangular with respect to the Milnor bases has to be B_{ArA} due to the unusual ordering $\dot{\prec}$ on B_{Mil} that is used for the proof. Since elements of B_{ArA} are monomials in the elements $\mathrm{Sq}(2^n)$, we can express an admissible monomial in the Milnor basis by using the product formula (2) for multiplying Milnor basis elements.

To convert a element from the Milnor basis to the basis of admissible monomials we show that the Arnon A basis is triangular with respect to the Milnor basis and define the γ and ordering $\dot{\prec}$ needed for the recursive formula of the form (1). For γ we make the following:

Definition 7.1. Let $\mathrm{Sq}\langle R\rangle = \mathrm{Sq}(r_1,\ldots,r_m)$ be a Milnor basis element. Define $\gamma(\mathrm{Sq}(r_1,\ldots,r_m)) = X_{k_0}^{n_0}X_{k_1}^{n_1}\cdots X_{k_p}^{n_p}$ where

1. $(n_0,k_0)\prec_L (n_1,k_1)\prec_L \cdots\prec_L (n_p,k_p)$ and
2. X_k^n is a factor of $X_{k_0}^{n_0}X_{k_1}^{n_1}\cdots X_{k_p}^{n_p}$ if and only if $\alpha_k(r_{n-k+1}) = 1$.

A heuristic for easily computing this gamma is very similar to that used for B_{PR}. First, write down the binary chart for $\mathrm{Sq}(r_1,\ldots,r_m)$. Then for each chart location (i,j) where there is a 1, we have an associated factor X_j^{i+j-1} of $\gamma\,\mathrm{Sq}(r_1,\ldots,r_m)$. These factors are then multiplied in the correct order.

For example, to compute $\gamma\,\mathrm{Sq}(2,5,1)$ we make the chart:

$$\begin{array}{ccc} & 1 & \\ 1 & 0 & \\ 0 & 1 & 1 \end{array}$$

and read off the factors X_1^1, X_0^1, X_2^3, and X_0^2. Multiplying them in the correct order we get $\gamma\,\mathrm{Sq}(2,5,1) = X_0^1X_1^1X_0^2X_2^3$.

The order $\dot{\prec}$ on B_{Mil} which we require is quite unusual. We begin with an ordering on pairs of integers.

Definition 7.2. Define an ordering $\ll$ on $\mathbb{N}\times\mathbb{N}$ by $(a,b)\ll(c,d)$ if

1. $a+b<c+d$ or
2. $a+b=c+d$ and $b<d$.

For example, $(0,0)$ is the smallest element in this ordering and the ordering begins with

$$(0,0)\ll(1,0)\ll(0,1)\ll(2,0)\ll(1,1)\ll(0,2)\ll(3,0)\ll\cdots.$$

The purpose of this ordering is to order the entries on our binary charts which will then provide an ordering on B_{Mil}.

Definition 7.3. Let $\mathrm{Sq}(r_1,\ldots,r_m)$ and $\mathrm{Sq}(s_1,\ldots,s_n)$ be elements of B_{Mil}. We say $\mathrm{Sq}(s_1,\ldots,s_n) \prec_A \mathrm{Sq}(r_1,\ldots,r_m)$ if there exists (h,k) such that
1. $\alpha_i(r_j) = \alpha_i(s_j)$ for all $(i,j) \ll (h,k)$ and
2. $\alpha_k(r_h) < \alpha_k(s_h)$.

In other words, we compare the entries of the binary charts of $\mathrm{Sq}(r_1,\ldots,r_m)$ and $\mathrm{Sq}(s_1,\ldots,s_n)$ in increasing $\ll$ order until we find the first location (h,k) where they differ. Whichever element has the 0 at (h,k) is the larger element. (Note that the second condition is equivalent to the condition $\alpha_k(r_h) = 0$ and $\alpha_k(s_h) = 1$.)

Armed with this γ and ordering $\prec_A$ on B_{Mil} we can now prove:

Theorem 7.1. *$\mathrm{Sq}(r_1,\ldots,r_m)$ is the $\prec_A$-largest summand of $\gamma\,\mathrm{Sq}(r_1,\ldots,r_m)_{\mathrm{Mil}}$.*

Thus the Arnon A basis is triangular with respect to the Milnor basis and we have the recursive formula (1) for converting an element of A from the Milnor basis to the basis B_{ArA}.

Corollary 7.2. *Let $\mathrm{Sq}\langle R\rangle \in B_{\mathrm{Mil}}$ and suppose $\gamma(\mathrm{Sq}\langle R\rangle)_{\mathrm{Mil}} = \mathrm{Sq}\langle R\rangle + \sum_i \mathrm{Sq}\langle R_i\rangle$. Then*

$$\mathrm{Sq}\langle R\rangle_{\mathrm{ArA}} = \gamma\,\mathrm{Sq}\langle R\rangle + \textstyle\sum_i \mathrm{Sq}\langle R_i\rangle_{\mathrm{ArA}}$$

is a well-defined recursive formula for computing $\mathrm{Sq}\langle R\rangle_{\mathrm{ArA}}$.

For example, to compute $\mathrm{Sq}(2,2)_{\mathrm{ArA}}$ we first compute $\gamma\,\mathrm{Sq}(2,2) = X_1^1 X_1^2$. By the Milnor product formula

$$\begin{aligned} X_1^1 X_1^2 &= \mathrm{Sq}(2)\,\mathrm{Sq}(4)\,\mathrm{Sq}(2) \\ &= \mathrm{Sq}(2,2) + \mathrm{Sq}(5,1). \end{aligned}$$

Applying Corollary 7.2 to the error term we find $\gamma\,\mathrm{Sq}(5,1) = X_0^0 X_0^1 X_2^2$. So by the product formula

$$\begin{aligned} X_0^0 X_0^1 X_2^2 &= \mathrm{Sq}(1)\,\mathrm{Sq}(2)\,\mathrm{Sq}(1)\,\mathrm{Sq}(4) \\ &= \mathrm{Sq}(5,1). \end{aligned}$$

Thus we have the desired answer

$$\mathrm{Sq}(2,2) = X_1^1 X_1^2 + X_0^0 X_0^1 X_2^2.$$

In order to prove these results we begin by defining some notation that will be convenient.

Definition 7.4. Let $\mathrm{Sq}(r_1,\ldots,r_m)$ be any Milnor basis element. We say $\mathrm{Sq}(r_1,\ldots,r_m)$ is *zero up to* (h,k) if $\alpha_i(r_j) = 0$ for all $(i,j) \ll (h,k)$.

Definition 7.5. Let $\mathrm{Sq}(r_1,\ldots,r_m)$ be any Milnor basis element. We say $\mathrm{Sq}(r_1,\ldots,r_m)$ has a 1 at (h,k) if $\alpha_k(r_h) = 1$.

Clearly if $\mathrm{Sq}\langle R\rangle$ is zero through (h,k) and $\mathrm{Sq}\langle S\rangle$ is not, then $\mathrm{Sq}\langle S\rangle \prec_A \mathrm{Sq}\langle R\rangle$. This notation is very intuitive when considering the ordering $\prec_A$ and the binary charts of Milnor basis elements, as in the following technical lemma.

Lemma 7.3. *Let* $\theta = X_{k_0}^{n_0} X_{k_1}^{n_1} \cdots X_{k_p}^{n_p} \in B_{\mathrm{ArA}}$ *and let* $\mathrm{Sq}(r_1,\ldots,r_m) = \gamma^{-1}(\theta)$. *Then*
1. $\mathrm{Sq}(r_1,\ldots,r_m)$ *has a* 1 *at* (n_0-k_0+1,k_0).
2. $\mathrm{Sq}(r_1,\ldots,r_m)$ *is zero up to* (n_0-k_0+1,k_0)
3. $X_{k_0}^{n_0} X_{k_1}^{n_1} \cdots X_{k_p}^{n_p} = \mathrm{Sq}(2^{n_0}) X_{k_0}^{n_0-1} X_{k_1}^{n_1} \cdots X_{k_p}^{n_p}$ *and* $X_{k_0}^{n_0-1} X_{k_1}^{n_1} \cdots X_{k_p}^{n_p} \in B_{\mathrm{ArA}}$ *(take* $X_{k_0}^{n_0-1}=1$ *if* $n_0=k_0$*).*
4. $\gamma^{-1}(X_{k_0}^{n_0-1} X_{k_1}^{n_1} \cdots X_{k_p}^{n_p}) = \mathrm{Sq}\left(r_1,\ldots,r_{h-1}, r_h+2^{k_0}, r_{h+1}-2^{k_0}, r_{h+2},\ldots,r_m\right)$ *where* $h = n_0-k_0$ *(if* $h=0$ *we interpret the right hand expression as* $\mathrm{Sq}\left(r_1-2^{k_0}, r_2,\ldots,r_m\right)$*).*

Proof. (1) By definition of γ, $X_{k_0}^{n_0}$ being a factor of θ implies that $\alpha_{k_0}\left(r_{n_0-k_0+1}\right)=1$ which in turn implies that $\mathrm{Sq}(r_1,\ldots,r_m)$ has a 1 at (n_0-k_0+1,k_0).

(2) Assume the contrary. Then there must be $(i,j) \ll (n_0-k_0+1,k_0)$ such that $\mathrm{Sq}(r_1,\ldots,r_m)$ has a 1 at (i,j), i.e. such that $\alpha_j(r_i)=1$. Thus by definition of γ, X_j^{i+j-1} must be a factor of θ. Now $(i,j) \ll (n_0-k_0+1,k_0)$ implies that either $i+j<n_0+1$ or else $i+j=n_0+1$ and $j<k_0$ so that in either case $(i+j-1,j) \prec_L (n_0,k_0)$. But this contradicts the factor $X_{k_0}^{n_0}$ must be the smallest factor in left lexicographic order of its indices by definition of B_{ArA}.

(3) This follows trivially from

$$(n_0-1,k_0) \prec_L (n_0,k_0) \prec_L \cdots \prec_L (n_m,k_m)$$

and $X_{k_0}^{n_0} = \mathrm{Sq}(2^{n_0})\,\mathrm{Sq}\left(2^{n_0-1}\right)\cdots \mathrm{Sq}\left(2^{k_0}\right) = \mathrm{Sq}(2^{n_0})\,X_{k_0}^{n_0-1}$.

(4) Since $X_{k_0}^{n_0} X_{k_1}^{n_1} \cdots X_{k_p}^{n_p}$ and $X_{k_0}^{n_0-1} X_{k_1}^{n_1} \cdots X_{k_p}^{n_p}$ only differ in the first factor, then by definition of γ, $\mathrm{Sq}(r_1,\ldots,r_m)$ and $\gamma^{-1}(X_{k_0}^{n_0-1} X_{k_1}^{n_1} \cdots X_{k_p}^{n_p})$ must have identical binary charts with the exception of the 1's corresponding to the leading factors. By (1), $\mathrm{Sq}(r_1,\ldots,r_m)$ has a 1 at (n_0-k_0+1,k_0) and $\gamma^{-1}(X_{k_0}^{n_0-1} X_{k_1}^{n_1} \cdots X_{k_p}^{n_p})$ has a 1 at (n_0-k_0,k_0). But by (2) $\mathrm{Sq}(r_1,\ldots,r_m)$ does not have a 1 at (n_0-k_0,k_0) and it is clear that $\gamma^{-1}(X_{k_0}^{n_0-1} X_{k_1}^{n_1} \cdots X_{k_p}^{n_p})$ does not have a 1 at (n_0-k_0+1,k_0) since $X_{k_0}^{n_0}$ is not a factor (remembering that $(n_0,k_0) \prec_L (n_1,k_1)$). Thus to obtain $\gamma^{-1}(X_{k_0}^{n_0-1} X_{k_1}^{n_1} \cdots X_{k_p}^{n_p})$ from $\mathrm{Sq}(r_1,\ldots,r_m)$ we simply remove the 1 at (n_0-k_0+1,k_0) by subtracting 2^{k_0} from $r_{n_0-k_0+1}$ and create a 1 at (n_0-k_0,k_0) by adding 2^{k_0} to $r_{n_0-k_0}$. Thus

$$\gamma^{-1}(X_{k_0}^{n_0-1} X_{k_1}^{n_1} \cdots X_{k_p}^{n_p}) = \mathrm{Sq}\left(r_1,\ldots,r_{n_0-k_0}+2^{k_0}, r_{n_0-k_0+1}-2^{k_0},\ldots,r_m\right)$$

as required. □

We are now ready to prove Theorem 7.1.

Proof of Theorem 7.1. It suffices to show that $\gamma^{-1}(X_{k_0}^{n_0} X_{k_1}^{n_1} \cdots X_{k_p}^{n_p})$is the $\prec_A$-largest summand of $(X_{k_0}^{n_0} X_{k_1}^{n_1} \cdots X_{k_p}^{n_p})_{\mathrm{Mil}}$. So let $X_{k_0}^{n_0} X_{k_1}^{n_1} \cdots X_{k_p}^{n_p} \in B_{\mathrm{ArA}}$. Then $\gamma^{-1}(X_{k_0}^{n_0} X_{k_1}^{n_1} \cdots X_{k_p}^{n_p})=\mathrm{Sq}(r_1,\ldots,r_m)$ where $\alpha_k(r_{n-k+1})=1$ if and only if X_k^n is a factor of $X_{k_0}^{n_0} X_{k_1}^{n_1} \cdots X_{k_p}^{n_p}$.

Let $q=\sum_{i=0}^{p}(n_i-k_i+1)$ which is the total number of factors of the form $\mathrm{Sq}(2^i)$ in the product $X_{k_0}^{n_0}X_{k_1}^{n_1}\cdots X_{k_p}^{n_p}$ when expanded using the definition of X_k^n. We proceed by induction on q.

If $q=1$ then $p=0$ and $n_0=k_0$ so that $(X_{k_0}^{n_0}X_{k_1}^{n_1}\cdots X_{k_p}^{n_p})_{\mathrm{Mil}}=\mathrm{Sq}\,(2^{n_0})_{\mathrm{Mil}}=\mathrm{Sq}\,(2^{n_0})$ and $\gamma^{-1}\left(X_{n_0}^{n_0}\right)=\mathrm{Sq}\,(2^{n_0})$ and hence the theorem holds.

Assume that the theorem is true for all θ in B_{ArA} having less than q factors of the form $\mathrm{Sq}\,(2^i)$. Then by Lemma 7.3

$$\gamma(X_{k_0}^{n_0-1}X_{k_1}^{n_1}\cdots X_{k_p}^{n_p})=\mathrm{Sq}\,(r_1,\ldots,r_h+2^{k_0},r_{h+1}-2^{k_0},\ldots,r_m)$$

where $h=n_0-k_0$. For brevity let

$$\tilde{R}=r_1,\ldots,r_h+2^{k_0},r_{h+1}-2^{k_0},\ldots,r_m.$$

So by our inductive hypothesis we have

$$\begin{aligned}\left(X_{k_0}^{n_0}X_{k_1}^{n_1}\cdots X_{k_p}^{n_p}\right)_{\mathrm{Mil}}&=\left(\mathrm{Sq}\,(2^{n_0})X_{k_0}^{n_0-1}X_{k_1}^{n_1}\cdots X_{k_p}^{n_p}\right)_{\mathrm{Mil}}\\&=\mathrm{Sq}\,(2^{n_0})\left(\mathrm{Sq}\,\langle\tilde{R}\rangle+\sum\mathrm{Sq}\langle S_i\rangle\right)\\&=\mathrm{Sq}\,(2^{n_0})\,\mathrm{Sq}\,\langle\tilde{R}\rangle+\sum\mathrm{Sq}\,(2^{n_0})\,\mathrm{Sq}\langle S_i\rangle,\end{aligned}$$

where $\mathrm{Sq}\langle S_i\rangle\prec_A\mathrm{Sq}\,\langle\tilde{R}\rangle$ for all i.

Now the matrix X

$$\begin{matrix}* & r_1 & \cdots & r_h & r_{h+1}-2^{k_0} & \cdots & r_m\\ 0 & 0 & \cdots & 2^{k_0} & 0 & \cdots & 0\end{matrix}\tag{12}$$

is $\mathrm{Sq}\,(2^{n_0})\,\mathrm{Sq}\,\langle\tilde{R}\rangle$-admissible since it clearly satisfies (3), (4) (because $2^h2^{k_0}=2^{n_0-k_0}2^{k_0}=2^{n_0}$), and (5) (since $2^{k_0}\in r_{h+1}$ by Lemma 7.3 it follows that $2^{k_0}\notin r_{h+1}-2^{k_0}$). The matrix X produces $\mathrm{Sq}\,(r_1,\ldots,r_m)$ as desired.

Let X be any other $\mathrm{Sq}\,(2^{n_0})\ \mathrm{Sq}\,\langle\tilde{R}\rangle$-allowable matrix which produces $\mathrm{Sq}\,(t_1,\ldots,t_n)$. Then X has the form

$$\begin{matrix}* & x_1 & x_2 & \cdots & x_w\\ y_0 & y_1 & y_2 & \cdots & y_w\end{matrix}\ .\tag{13}$$

We consider two cases: $h\neq0$ and $h=0$.

Case 1: $h\neq0$. Since X is admissible, $\sum2^iy_i=2^{n_0}$ and hence $y_h\leq2^{k_0}$. But $y_h\neq2^{k_0}$ since we are assuming this matrix is not the same as (12). So $y_h<2^{k_0}$. Thus by (3) $x_h=r_h+2^{k_0}-y_h$. But since $y_h<2^{k_0}$ there exists $u\leq k_0$ such that $2^u\in x_h$ (this follows from the fact that $2^i\notin r_h$ for $i\leq k_0$ since $\mathrm{Sq}\,(r_1,\ldots,r_m)$ is zero up to $(h+1,k_0)$ by Lemma 7.3). But also $t_h=x_h+y_{h-1}$ and so by (5) $2^u\in t_h$ also. Thus $\mathrm{Sq}\,(t_1,\ldots,t_n)$ has a 1 at (h,u). But $h+u\leq h+k_0<h+k_0+1$ so that $(h,u)\ll(h+1,k_0)$. But $\mathrm{Sq}\,(r_1,\ldots,r_m)$ is zero up to $(h+1,k_0)$ so that $\mathrm{Sq}\,(t_1,\ldots,t_n)\prec_A\mathrm{Sq}\,(r_1,\ldots,r_m)$.

Case 2: $h=0$. In this case $n_0=k_0$. Since X is admissible, $\sum2^iy_i=2^{n_0}$ and thus there must be some v such that $y_v\neq0$ and consequently some u such that $2^u\in y_v$ with $u+v\leq n_0$. Notice also that we have $u<k_0$ since we are assuming this matrix is

not the same as (12). By (5) $2^u \in y_v$ implies $2^u \in t_{v+1}$. Thus $\mathrm{Sq}(t_1,\dots,t_n)$ has a 1 at $(v+1,u)$. But $u+v+1 \le n_0+1$ and $u < k_0$ so that $(v+1,u) \ll (1,k_0)$. By Lemma 7.3 $\mathrm{Sq}(r_1,\dots,r_m)$ is zero up to $(1,k_0)$ so that $\mathrm{Sq}(t_1,\dots,t_n) \prec_A \mathrm{Sq}(r_1,\dots,r_m)$.

So in both cases we have shown that any other $\mathrm{Sq}(2^{n_0})\ \mathrm{Sq}\langle\tilde{R}\rangle$-allowable matrix other than (12) produces $\mathrm{Sq}(t_1,\dots,t_n)$ which is strictly $\prec_A$ less than $\mathrm{Sq}(r_1,\dots,r_m)$. Thus $\mathrm{Sq}(r_1,\dots,r_m)$ is a summand of the product $\mathrm{Sq}(2^{n_0})\mathrm{Sq}\langle\tilde{R}\rangle$.

So all that remains to be demonstrated is that $\mathrm{Sq}(r_1,\dots,r_m)$ is not a summand of $\mathrm{Sq}(2^{n_0})\mathrm{Sq}\langle S_i\rangle$ for any of the terms $\mathrm{Sq}\langle S_i\rangle \prec_A \mathrm{Sq}\langle\tilde{R}\rangle$. So let $\mathrm{Sq}\langle S_i\rangle = \mathrm{Sq}(s_1,\dots,s_n)$ be any such term. We again consider two cases.

Case 1: $h=0$. In this case $n_0=k_0$. Let X be a $\mathrm{Sq}(2^{n_0})\mathrm{Sq}\langle S_i\rangle$-allowable matrix (which must be of the form (13)). Since $\sum 2^i y_i = 2^{n_0}$ there must be some v such that $y_v \ne 0$ and consequently some u such that $2^u \in y_v$ with $u+v \le n_0$. By (5) $2^u \in y_v$ implies $2^u \in t_{v+1}$. Thus $\mathrm{Sq}(t_1,\dots,t_n)$ has a 1 at $(v+1,u)$. Since $u+v \le n_0$ it follows that $u+v+1 \le n_0+1$.

Case 1.1: $u+v+1 < n_0+1$ *or* $u+v+1 = n_0+1$ *and* $u < n_0$. In this case $(v+1,u) \ll (1,n_0)$. But $\mathrm{Sq}(r_1,\dots,r_m)$ is zero up to $(1,n_0)$ so that $\mathrm{Sq}(t_1,\dots,t_n) \prec_A \mathrm{Sq}(r_1,\dots,r_m)$.

Case 1.2: $u+v+1 = n_0+1$ *and* $u = n_0$. In this case $v=0$. Since X is allowable, $2^{n_0} \notin s_1$ (by (5)). Thus X produces $\mathrm{Sq}(t_1,\dots,t_n)$ where $t_1 = s_1 + 2^{n_0}$, and $t_i = s_i$ for $i>1$. Thus it is easy to see that the binary chart of $\mathrm{Sq}(t_1,\dots,t_n)$ is identical to that of $\mathrm{Sq}(s_1,\dots,s_n)$ with the exception of the 1 at location $(1,n_0)$ of $\mathrm{Sq}(t_1,\dots,t_n)$. Similarly, the binary chart of $\mathrm{Sq}(r_1,\dots,r_m)$ is identical to that of $\mathrm{Sq}\langle\tilde{R}\rangle$ with the exception of the 1 at location $(1,n_0)$ of $\mathrm{Sq}(r_1,\dots,r_m)$ (since $\mathrm{Sq}\langle\tilde{R}\rangle$ is zero up to $(n_1-k_1+1,k_1) \gg (1,k_0)$). Then the fact that $\mathrm{Sq}(s_1,\dots,s_n) \prec_A \mathrm{Sq}\langle\tilde{R}\rangle$ implies that there exists (a,b) such that the binary charts of $\mathrm{Sq}(s_1,\dots,s_n)$ and $\mathrm{Sq}\langle\tilde{R}\rangle$ match at all locations $(i,j) \ll (a,b)$ and that $\mathrm{Sq}(s_1,\dots,s_n)$ has a 1 at (a,b) while $\mathrm{Sq}\langle\tilde{R}\rangle$ has a 0 at (a,b). Simply changing the 0 at $(1,n_0)$ on both charts to a 1 does not affect this situation so that once again $\mathrm{Sq}(t_1,\dots,t_n) \prec_A \mathrm{Sq}(r_1,\dots,r_m)$.

Case 2: $h>0$. Let X be a $\mathrm{Sq}(2^{n_0})\mathrm{Sq}\langle S_i\rangle$-allowable matrix which produces $\mathrm{Sq}(t_1,\dots,t_n)$ (and must be of the form (13)). Once again since $\sum 2^i y_i = 2^{n_0}$ there must be some v such that $y_v \ne 0$ and consequently some u such that $2^u \in y_v$ with $u+v \le n_0$. By (5) $2^u \in y_v$ implies $2^u \in t_{v+1}$. Thus $\mathrm{Sq}(t_1,\dots,t_n)$ has a 1 at $(v+1,u)$.

Case 2.1: $u+v < n_0$ *or* $(u+v = n_0$ *and* $u < k_0)$. In this case $(v+1,u) \ll (n_0-k_0+1,k_0)$. But $\mathrm{Sq}(r_1,\dots,r_m)$ is zero up to (n_0-k_0+1,k_0) so that $\mathrm{Sq}(t_1,\dots,t_n) \prec_A \mathrm{Sq}(r_1,\dots,r_m)$.

Case 2.2: $u+v = n_0$ *and* $u \ge k_0$. Then X has the form

$$\begin{matrix} * & s_1 & \cdots & s_{v-1} & s_v - 2^u & s_{v+1} & \cdots & s_n \\ 0 & 0 & \cdots & 0 & 2^u & 0 & \cdots & 0 \end{matrix}$$

Case 2.2.1: $2^u \notin s_v$. In this case $2^u \in s_v - 2^u$ which implies that $2^u \in t_v$. Thus $\mathrm{Sq}(t_1,\dots,t_n)$ has a 1 at (v,u). But $u+v=n_0 < n_0+1$ so that $(v,u) \ll (n_0-k_0+1,k_0)$. But $\mathrm{Sq}(r_1,\dots,r_m)$ is zero up to (n_0-k_0+1,k_0) so that $\mathrm{Sq}(t_1,\dots,t_n) \prec_A \mathrm{Sq}(r_1,\dots,r_m)$.

Case 2.2.2: $2^u \in s_v$. In this case we have

$$\mathrm{Sq}(t_1,\dots,t_n) = \mathrm{Sq}(s_1,\dots,s_v - 2^u, s_{v+1} + 2^u, s_{v+2},\dots,s_n).$$

Notice that $2^u \notin s_{v+1}$ since X is admissible, so that the only difference between the binary charts of $\mathrm{Sq}(t_1,\dots,t_n)$ and $\mathrm{Sq}(s_1,\dots,s_n)$ is that the 1 at (v,u) in $\mathrm{Sq}(s_1,\dots,s_n)$ is moved to location $(v+1,u)$ in $\mathrm{Sq}(t_1,\dots,t_n)$. Also the difference between the binary charts of $\mathrm{Sq}\langle\tilde{R}\rangle$ and $\mathrm{Sq}(r_1,\dots,r_m)$ is that the 1 at location (h,k_0) in $\mathrm{Sq}\langle\tilde{R}\rangle$ is moved to location $(h+1,k_0)$ in $\mathrm{Sq}(r_1,\dots,r_m)$. By definition $\mathrm{Sq}(s_1,\dots,s_n) \prec_A \mathrm{Sq}\langle\tilde{R}\rangle$ implies that there exists (a,b) such that the binary charts of $\mathrm{Sq}(s_1,\dots,s_n)$ and $\mathrm{Sq}\langle\tilde{R}\rangle$ match at all locations $(i,j) \ll (a,b)$ and that $\mathrm{Sq}(s_1,\dots,s_n)$ has a 1 at (a,b) while $\mathrm{Sq}\langle\tilde{R}\rangle$ has a 0 at (a,b).

Case 2.2.2.1: $u = k_0$. In this case $(v,u) = (h,k_0)$. It is clear that simply moving the 1 at (h,k_0) to location $(h+1,k_0)$ on both binary charts to a 1 does not affect the fact that (a,b) is the first location where the charts differ and does not change the values of the charts at (a,b), so that once again $\mathrm{Sq}(t_1,\dots,t_n) \prec_A \mathrm{Sq}(r_1,\dots,r_m)$.

Case 2.2.2.2: $u > k_0$. Since $\mathrm{Sq}\langle\tilde{R}\rangle$ has a 1 at (h,k_0) and is zero up to (h,k_0) and also $\mathrm{Sq}\langle S_i\rangle \prec_A \mathrm{Sq}\langle\tilde{R}\rangle$ then either $\mathrm{Sq}(s_1,\dots,s_n)$ is not zero up to (h,k_0) or else it is and it has a 1 at (h,k_0) also.

Case 2.2.2.2.1: $\mathrm{Sq}(s_1,\dots,s_n)$ is not zero up to (h,k_0). In this case there is a 1 at (i,j) for some $(i,j) \ll (h,k_0) \ll (v,u)$. As the binary charts of $\mathrm{Sq}(t_1,\dots,t_n)$ and $\mathrm{Sq}(s_1,\dots,s_n)$ only differ at locations (v,u) and $(v+1,u)$, $\mathrm{Sq}(t_1,\dots,t_n)$ must also have a 1 at $(i,j) \ll (h,k_0) \ll (h+1,k_0)$. Thus since $\mathrm{Sq}(r_1,\dots,r_m)$ is zero up to $(h+1,k_0)$ we have $\mathrm{Sq}(t_1,\dots,t_n) \prec_A \mathrm{Sq}(r_1,\dots,r_m)$.

Case 2.2.2.2.2: $\mathrm{Sq}(s_1,\dots,s_n)$ *is zero up to* (h,k_0) *and has a* 1 *at* (h,k_0). Since the binary charts of $\mathrm{Sq}(t_1,\dots,t_n)$ and $\mathrm{Sq}(s_1,\dots,s_n)$ only differ at locations (v,u) and $(v+1,u)$ and $(h,k_0) \ll (v,u)$, $\mathrm{Sq}(t_1,\dots,t_n)$ must also have a 1 at $(h,k_0) \ll (h+1,k_0)$. Thus since $\mathrm{Sq}(r_1,\dots,r_m)$ is zero up to $(h+1,k_0)$ we have $\mathrm{Sq}(t_1,\dots,t_n) \prec_A \mathrm{Sq}(r_1,\dots,r_m)$. □

8. Non-triangular bases

The remaining bases, B_{Wall}, B_{WdY}, and B_{WdZ} are not triangular with respect to the Milnor basis. There is an interesting relationship between the B_{Wall} and B_{WdZ} bases however, which we note in this section.

To see that B_{Wall} is not triangular with respect to the Milnor basis we consider grading 9. In this grading the elements of B_{Wall} are $\mathrm{Sq}^{8,1}$, $\mathrm{Sq}^{1,2,4,2}$, $\mathrm{Sq}^{2,4,1,2}$, $\mathrm{Sq}^{2,4,2,1}$, and $\mathrm{Sq}^{4,2,1,2}$. By the Milnor product formula these equal:

$$\mathrm{Sq}^{8,1} = \mathrm{Sq}(9) + \mathrm{Sq}(6,1),$$
$$\mathrm{Sq}^{1,2,4,2} = \mathrm{Sq}(3,2),$$
$$\mathrm{Sq}^{2,4,1,2} = \mathrm{Sq}(6,1) + \mathrm{Sq}(0,3) + \mathrm{Sq}(3,2),$$

$$\mathrm{Sq}^{2,4,2,1} = \mathrm{Sq}\,(3,2) + \mathrm{Sq}\,(0,3) + \mathrm{Sq}\,(2,0,1),$$
$$\mathrm{Sq}^{4,2,1,2} = \mathrm{Sq}\,(6,1) + \mathrm{Sq}\,(0,3) + \mathrm{Sq}\,(2,0,1).$$

Clearly, any bijection γ mapping B_{Wall} to B_{Mil} must have $\gamma\,\mathrm{Sq}^{1,2,4,2} = \mathrm{Sq}\,(3,2)$. Now suppose we want to find an ordering $\prec$ of B_{Mil} in grading 9 and extend γ so that $\theta_{\mathrm{Mil}} = \gamma(\theta) + \sum \mathrm{Sq}\,\langle R_i\rangle$ where each $\mathrm{Sq}\,\langle R_i\rangle \prec \gamma(\theta)$. Then among the elements $\mathrm{Sq}\,(6,1)$, $\mathrm{Sq}\,(0,3)$, and $\mathrm{Sq}\,(2,0,1)$ we must decide which element is greatest in terms of $\prec$. Suppose we choose $\mathrm{Sq}\,(6,1)$ to be the largest. Then the condition that γ map θ to the largest summand forces $\gamma\,\mathrm{Sq}^{2,4,1,2} = \gamma\,\mathrm{Sq}^{4,2,1,2} = \mathrm{Sq}\,(6,1)$ which contradicts the injectivity of γ. A similar argument shows that we cannot choose either $\mathrm{Sq}\,(0,3)$ or $\mathrm{Sq}\,(2,0,1)$ for the $\prec$ largest element. Thus no such ordering and gamma exist, and we conclude B_{Wall} is not triangular with respect tot the Milnor basis. An exactly analogous argument in grading 9 proves that both B_{WdY} and B_{WdZ} are not triangular with respect to the Milnor basis either.

The Wood bases are related to each other in the same sense that the P_t^s bases described above are: one basis can be obtained from the other by simply changing the order of the factors in the monomials. There also is an interesting relationship of sorts between the Wall basis and the Wood Z basis. We have the following:

Theorem 8.1. *Y_{n-k}^k is the $\prec_E$ largest summand of $(Q_k^n)_{\mathrm{Mil}}$.*

The proof of this theorem is very similar to the proof of Theorem 5.1 and will not be presented here. Thus we are naturally led to consider the bijection $\gamma : B_{\mathrm{Wall}} \to B_{\mathrm{WdZ}}$ by

$$\gamma\left(Q_{k_0}^{n_0} Q_{k_1}^{n_1} \cdots Q_{k_p}^{n_p}\right) = Y_{n_0-k_0}^{k_0} Y_{n_1-k_1}^{k_1} \cdots Y_{n_p-k_p}^{k_p}.$$

It is a simple matter to verify that the order of the factors is such that the right hand side is indeed an element of B_{WdZ} as claimed.

We close this section by commenting that it is conceivable that these three bases are triangular with respect to one another, but knowing this would not provide us with a recursive change of basis formula of the form (1) since this relies on the Milnor product formula to convert from the given basis to the Milnor basis, and we have no analogous product formula for these bases.

9. Product relations

In order to improve on the change of basis formulas derived above, we would like to obtain explicit non-recursive formulas. As a first step in this direction it would be desirable to know which elements are common to two given bases. For example, it is well known that the elements $\mathrm{Sq}(n)$ are common to both the Milnor and admissible monomial bases. But are these the only such elements? The answer is no, and further investigation yields an infinite subset of $B_{\mathrm{Mil}} \cap B_{\mathrm{Adm}}$. By Theorem 4.1 any element $\theta \in B_{\mathrm{Mil}} \cap B_{\mathrm{Adm}}$ must satisfy $\gamma(\theta) = \theta$, i.e. it must be an eigenvector of γ (extended to a linear transformation of A).

Theorem 9.1. *If* $r_i \equiv -1 \mod 2^{\omega(r_{i+1})}$ *for all* $1 \leq i < m$ *then* $\mathrm{Sq}(r_1,\ldots,r_m) \in B_{\mathrm{Mil}} \cap B_{\mathrm{Adm}}$ *(and in this case* $\mathrm{Sq}(r_1,\ldots,r_m) = \gamma\mathrm{Sq}(r_1,\ldots,r_m)$*).*

We point out that this linear algebra result is also providing us with information about products, i.e. $\mathrm{Sq}^{t_1}\mathrm{Sq}^{t_2}\cdots\mathrm{Sq}^{t_m} = \mathrm{Sq}(r_1,\ldots,r_m)$ where $r_i = t_i - 2t_{i+1}$ (take $t_{m+1} = 0$) if $r_i \equiv -1 \mod 2^{\omega(r_{i+1})}$ for all $1 \leq i < m$.

We also note out that the condition $r_i \equiv -1 \mod 2^{\omega(r_{i+1})}$ can easily be checked by writing the ordinary binary representations of numbers $r_1, r_2, \ldots, r_m$ in horizontally above one another (with r_1 on top) and checking that no digit ever appears below a 0. This is because of the following trivial fact which we state without proof:

$$r \equiv -1 \mod 2^w \Leftrightarrow 2^k \in r \quad \text{for all } k < w. \tag{14}$$

For example, $\mathrm{Sq}(13,5,1)$ is not equal to an admissible monomial because writing the indices in base 2 yields:

$$\begin{array}{rcrrrr} 13 & = & 1 & 1 & 0 & 1_2 \\ 5 & = & & 1 & 0 & 1_2 \\ 1 & = & & & & 1_2 \end{array}$$

and the 0 in the two's column of the 5 is beneath the 0 in the same column for 13. On the other hand, $\mathrm{Sq}(7,5,1)$ does satisfy the required condition and so by Theorem 9.1 we deduce that $\mathrm{Sq}(7,5,1) = \mathrm{Sq}^{21}\mathrm{Sq}^{7}\mathrm{Sq}^{1}$.

We will also need to make use of the following fact whose verification is an elementary exercise in binary arithmetic.

Lemma 9.2. *Let* x, y, r, w *be non-negative integers. If* $r \equiv -1 \mod 2^w$ *and* $x + y = r$ *then for any* $k < w$ *either* $2^k \in x$ *or* $2^k \in y$ *but not both, i.e.* $\alpha_k(x) + \alpha_k(y) = 1$.

We now turn our attention to proving Theorem 9.1.

Proof of Theorem 9.1. Let $R = r_1,\ldots,r_m$ be any sequence satisfying $r_i \equiv -1 \mod 2^{\omega(r_{i+1})}$ for all i. We would like to show that $\gamma\,\mathrm{Sq}\langle R\rangle = \mathrm{Sq}\langle R\rangle$. We proceed by induction on m.

If $m = 1$ then $\gamma\,\mathrm{Sq}(r_1) = \mathrm{Sq}(r_1)$ by definition of γ.

Now for the inductive hypothesis assume that for any $\mathrm{Sq}(s_1,\ldots,s_k) \in B_{\mathrm{Mil}}$ with $k < m$, if $s_i \equiv -1 \mod 2^{\omega(s_{i+1})}$ for all i then $\gamma\,\mathrm{Sq}\langle S\rangle = \mathrm{Sq}\langle S\rangle$. In particular, we have $\gamma\,\mathrm{Sq}(r_2,\ldots,r_m) = \mathrm{Sq}(r_2,\ldots,r_m)$.

Let $\mathrm{Sq}^{\langle T\rangle} = \gamma(R)$ where $T = t_1,\ldots,t_m$. Then clearly $\gamma\mathrm{Sq}(r_2,\ldots,r_m) = \mathrm{Sq}^{t_2}\cdots\mathrm{Sq}^{t_m}$ so that

$$\begin{aligned} \gamma\,\mathrm{Sq}\langle R\rangle &= \mathrm{Sq}(t_1)\,\mathrm{Sq}(t_2)\cdots\mathrm{Sq}(t_m) \\ &= \mathrm{Sq}(t_1)\,\gamma\mathrm{Sq}(r_2,\ldots,r_m) \\ &= \mathrm{Sq}(t_1)\,\mathrm{Sq}(r_2,\ldots,r_m). \end{aligned}$$

So it suffices to show that $\mathrm{Sq}(t_1)\,\mathrm{Sq}(r_2,\ldots,r_m)=\mathrm{Sq}(r_1,\ldots,r_m)$ by the Milnor product formula.

Let X be a $\mathrm{Sq}(t_1)\,\mathrm{Sq}(r_2,\ldots,r_m)$-allowable matrix. Then X is of the form

$$\begin{matrix} * & x_2 & x_3 & \cdots & x_m \\ y_1 & y_2 & y_3 & \cdots & y_m \end{matrix}$$

such that for $2 \le i \le m$

$$x_i + y_i = r_i \tag{15}$$

$$x_i \asymp y_{i-1} \tag{16}$$

and also satisfying

$$t_1 = \sum_{i=1}^{m} 2^{i-1}\, y_i. \tag{17}$$

Let $j > 2$ and suppose $2^i \in x_j$. We would like to show that $2^i \in x_{j-1}$ also. Now $2^i \in x_j$ implies that $2^i \le x_j \le r_j < 2^{\omega(r_j)}$ by (15) and $r_{j-1} \equiv -1 \mod 2^{\omega(r_j)}$ implies that $2^k \in r_{j-1}$ for all $k < \omega(r_j)$ by (14). Combining these facts shows $2^i \in r_{j-1}$. So by (15) and Lemma 9.2 either $2^i \in x_{j-1}$ or $2^i \in y_{j-1}$. But $2^i \notin y_{j-1}$ by condition (16), so $2^i \in x_{j-1}$. Thus we have shown that $2^i \in x_j$ implies $2^i \in x_{j-1}$. So by induction we have $2^i \in x_j$ implies $2^i \in x_2$. In particular, $\nu(x_j) \ge \nu(x_2)$ for all j.

Now suppose $x_2 \ne 0$. Then $2^{\nu(x_2)} \in x_2$. By Lemma 9.2 this implies that $2^{\nu(x_2)} \in r_2$ which in turn implies that $2^{\nu(x_2)} \in r_1$ by (14). Now since $\nu(x_j) \ge \nu(x_2)$ if follows that $2^{\nu(x_2)}$ divides x_j for all j, i.e. that

$$x_j = 2^{\nu(x_2)} h_j \tag{18}$$

for some non-negative integer h_j. Solving (17) for y_1, substituting for t_1 using (8) and applying (18) gives us

$$\begin{aligned} y_1 &= t_1 - \sum_{k=2}^{m} 2^{k-1} y_k \\ &= \sum_{k=1}^{m} 2^{k-1} r_k - \sum_{k=2}^{m} 2^{k-1} y_k \\ &= r_1 + \sum_{k=2}^{m} 2^{k-1} (r_k - y_k) \\ &= r_1 + \sum_{k=2}^{m} 2^{k-1} x_k \end{aligned}$$

$$= r_1 + \sum_{k=2}^{m} 2^{k-1} 2^{v(x_2)} h_k$$

$$= r_1 + 2^{v(x_2)+1} \left(\sum_{k=2}^{m} 2^{k-2} h_k \right).$$

Combining this with the fact that $2^{v(x_2)} \in r_1$, it follows by (7) that $2^{v(x_2)} \in y_1$. Thus $2^{v(x_2)} \in y_1$ and $2^{v(x_2)} \in x_2$ which contradicts (16). Therefore our assumption that $x_2 \neq 0$ must be false.

So $x_2 = 0$. But $v(x_j) \geq v(x_2)$ for all j, so it follows that $x_j = 0$ for all j. Hence X must be the matrix

$$\begin{matrix} * & 0 & 0 & \cdots & 0 \\ r_1 & r_2 & r_3 & \cdots & r_m \end{matrix}$$

which is clearly admissible and produces $\mathrm{Sq}(r_1, \ldots, r_m)$. □

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